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N-dimensional integrability from two-photon coalgebra symmetry

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Abstract

A wide class of Hamiltonian systems with N degrees of freedom and endowed with, at least, $(N-2)$ functionally independent integrals of motion in involution is constructed by making use of the two-photon Lie–Poisson coalgebra (h_6, Δ). The set of $(N-2)$ constants of the motion is shown to be a universal one for all these Hamiltonians, irrespective of the dependence of the latter on several arbitrary functions and N free parameters. Within this large class of quasi-integrable N-dimensional Hamiltonians, new families of completely integrable systems are identified by finding explicitly a new independent integral $\mathcal{I}$ through the analysis of the sub-coalgebra structure of h_6. In particular, new completely integrable N-dimensional Hamiltonians describing natural systems, geodesic flows and static electromagnetic Hamiltonians are presented.

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1. Introduction

Due to its physical and mathematical relevance, the construction of completely integrable Hamiltonian systems focuses on intense research activity that makes use of many different approaches and techniques (see, for instance, [1–4]). However, the number of known integrable systems (in the Liouville sense [5]) that can be generalized for an arbitrary number N of degrees of freedom is relatively scarce. In most of the known cases, such an N-dimensional (ND) integrability is based on some underlying symmetry that allows for the appropriate propagation of the integrability properties to arbitrary dimension (see the systems described in [6–22] and references therein).

The aim of this paper is the construction of new families of classical integrable Hamiltonian systems with N degrees of freedom, depending on several arbitrary functions and N free parameters, and whose integrals of the motion will be also explicitly given. These results will be obtained by applying the coalgebra symmetry method [23–27] to the Poisson version of the so-called two-photon/Schrödinger algebra h_6 [28–30]. As we shall show in the following, the h_6-coalgebra symmetry turns out to be extremely powerful, since the *N*-particle framework provided by the coalgebra structure encompasses different *N*-body symmetries that underlie several useful and explicit integrability properties. Moreover, one can easily realize that the approach presented here is also applicable in order to ensure the integrability of the quantum mechanical analogues of all the Hamiltonians contained in this paper.

Let us recall that the h_6 Lie–Poisson algebra is spanned by the six abstract generators $\{K, A_+, A_-, B_+, B_-, M\}$, whose one-particle symplectic realization is given by

$$\begin{array}{lll} A_+ = \lambda_1 p_1 & A_- = \lambda_1 q_1 & K = q_1 p_1 - \dfrac{\lambda_1^2}{2} \\ B_+ = p_1^2 & B_- = q_1^2 & M = \lambda_1^2, \end{array} \tag{1.1}$$

where λ_1 is a non-vanishing constant that labels the previous symplectic realization and where we have considered the usual Poisson bracket $\{q_1, p_1\} = 1$. The abstract Poisson brackets defining h_6 read as

$$\begin{array}{lll} \{K, A_+\} = A_+ & \{K, A_-\} = -A_- & \{A_-, A_+\} = M \\ \{K, B_+\} = 2B_+ & \{K, B_-\} = -2B_- & \{B_-, B_+\} = 4K + 2M \\ \{A_+, B_-\} = -2A_- & \{A_+, B_+\} = 0 & \{M, \cdot\} = 0 \\ \{A_-, B_+\} = 2A_+ & \{A_-, B_-\} = 0. & \end{array} \tag{1.2}$$

A direct inspection of this algebra makes its rich subalgebra structure evident (for instance, the $gl(2)$ subalgebra $\{K, B_+, B_-, M\}$, the oscillator one $h_4 \equiv \{K, A_+, A_-, M\}$ and the Heisenberg one $h_3 \equiv \{A_+, A_-, M\}$ can be easily identified). This is one of the main features of h_6, since this algebra generalizes many lower dimensional Lie symmetries in a transparent way, a fact that will be relevant in order to find new and more general integrability structures.

As a consequence of (1.1), any quadratic Hamiltonian with one degree of freedom can always be written as a linear combination of the h_6 generators in the above representation. This is indeed the origin of the quantum mechanical relevance of the h_6 Lie algebra as a dynamical symmetry, since the quantum counterpart of such a quadratic Hamiltonian can be interpreted as a single-mode radiation field Hamiltonian including the number operator $\hat{K}$, creation and annihilation operators $\hat{A}_+$ and $\hat{A}_-$, and two-photon creation and annihilation operators $\hat{B}_+$ and $\hat{B}_-$ (the generator $\hat{M}$ is a central one) respectively. In this context, different applications of the h_6 symmetry can be found in [29], and it is also interesting to recall that h_6 is isomorphic to the (1 + 1)D centrally extended Schrödinger Lie algebra [31]. In fact, this isomorphism provides two Casimir functions for h_6, which are the central generator M and the fourth-order Casimir [32]:

$$\mathcal{C} = \left(MB_+ - A_+^2\right)\left(MB_- - A_-^2\right) - (MK - A_-A_+ + M^2/2)^2, \tag{1.3}$$

that will play a relevant role hereafter. The function $\mathcal{C}$ can be reduced to a third-order invariant by extracting M as a common factor:

$$\mathcal{C}_{h_6} = \mathcal{C}/M = MB_+B_- - B_+A_-^2 - B_-A_+^2 - M(K + M/2)^2 + 2A_-A_+(K + M/2). \tag{1.4}$$

Note that if we substitute the symplectic realization (1.1) within the two Casimir functions, the former is characterized by the values $M = \lambda_1^2$ and $\mathcal{C}_{h_6} = 0$.

Now, if we endow the h_6 Poisson algebra with a primitive coalgebra structure, i.e. with a two-body coproduct (which is a Poisson algebra homomorphism $\Delta : h_6 \to h_6 \otimes h_6$ [24, 33]) given by

$$\Delta(X) = X \otimes 1 + 1 \otimes X \qquad X \in \{K, A_+, A_-, B_+, B_-, M\}, \tag{1.5}$$

then the one-particle dynamical symmetry given by (1.1) can be generalized for any number of degrees of freedom. As we shall see in section 2, this gives rise to an infinite family of N-particle Hamiltonians defined as *any* smooth function $\mathcal{H}$ of the generators of h_6

$$\mathcal{H} = \mathcal{H}(K, B_+, B_-, A_+, A_-, M)$$

and provided that $\mathcal{H}$ is realized in the N-particle symplectic realization coming from the coproduct structure. More importantly, the m-particle realizations $(m = 3, \ldots, N)$ of the coproducts of the Casimir function $\mathcal{C}_{h_6}$ will provide a set of $(N-2)$ independent integrals of the motion in involution with the whole family of Hamiltonians $\mathcal{H}$ (as we shall explain later, the $m = 2$ realization gives a vanishing constant of the motion). Moreover, since there is only *one* integral of motion left in order to get the *complete integrability* of $\mathcal{H}$, we shall name all these systems as *quasi-integrable* Hamiltonians. This construction will be described in detail in section 2, and some preliminary results can be found in [33]. Moreover, having in mind further applications, particular choices for $\mathcal{H}$ leading to ND natural systems, geodesic flows and static electromagnetic Hamiltonians will be explicitly identified.

We stress that in the coalgebra symmetry framework, the dimension of the coalgebra and the number of its nonlinear Casimirs are essential in order to analyse whether the coproduct gives either complete or partial integrability under a chosen symplectic realization. This dimensionality problem is essential for coalgebras with dimension greater than 3, and it has been fully analysed in [27]. In fact, the h_6 coalgebra provides the first explicit case of the coalgebra construction for a six-dimensional coalgebra with two Casimir functions in which one of them is a linear one (the generator M). Thus, the quasi-integrability is an essential feature of the h_6 coalgebra that could not appear in any of the examples of the coalgebra construction considered so far (see, for instance, [23–27, 34–36]) because all such systems come from three-dimensional coalgebras with one nonlinear Casimir, and their complete integrability is then guaranteed by construction.

Therefore, as a consequence of the quasi-integrability of the h_6-coalgebra symmetry, it becomes important to develop some additional algebraic machinery in order to investigate for which choices of the generic Hamiltonian $\mathcal{H}$ the complete integrability can be restored. This is the aim of sections 3–5, in which we present a novel algebraic completion of the coalgebra approach that provides a guide to obtaining the additional independent integral of motion $\mathcal{I}$ by exploiting the rich subalgebra structure of h_6. This approach is fully general and can be applied to any coalgebra with dimension greater than 3.

In particular, in section 3, we analyse two different possibilities in order to find an extra integral $\mathcal{I}$. We remark that, in both cases, $\mathcal{I}$ will always be defined as a function of the generators of h_6, since this guarantees its existence for any number of degrees of freedom. The first case arises if the chosen Hamiltonian can be defined on a subalgebra of h_6, since then the N-particle Casimir of such a subalgebra can be directly identified with the additional constant of motion $\mathcal{I}$. This possibility is discussed in section 4, where the rich subalgebra structure for h_6 is fully described. A second possibility is analysed in section 5, where we introduce five new families of ND completely integrable systems that have one of the five non-central generators of h_6 as the remaining integral of motion $\mathcal{I}$. This latter construction turns out to be powerful, since all these systems depend on several arbitrary functions and provide a large number of new instances of interesting Hamiltonians, from both the mathematical and the physical viewpoints. Finally, some comments and further research lines are presented in a concluding section.

2. An infinite family of quasi-integrable systems

The integrability properties of the generic N-particle Hamiltonians with h_6-coalgebra symmetry rely on the following result.

Theorem 1. *Let* $\{\mathbf{q}, \mathbf{p}\} = \{(q_1, \ldots, q_N), (p_1, \ldots, p_N)\}$ *be N pairs of canonical variables. The ND Hamiltonian*

$$\mathcal{H} = \mathcal{H}(K, B_+, B_-, A_+, A_-, M), \tag{2.1}$$

defined as any smooth function $\mathcal{H} : \mathbb{R}^6 \to \mathbb{R}$ *and*

$$\begin{aligned} A_+ &= \sum_{i=1}^N \lambda_i p_i \qquad A_- = \sum_{i=1}^N \lambda_i q_i \qquad K = \sum_{i=1}^N \left(q_i p_i - \frac{\lambda_i^2}{2} \right) \\ B_+ &= \sum_{i=1}^N p_i^2 \qquad B_- = \sum_{i=1}^N q_i^2 \qquad M = \sum_{i=1}^N \lambda_i^2, \end{aligned} \tag{2.2}$$

where λ_i *are N arbitrary parameters, is quasi-integrable. The* $(N-2)$ *functionally independent integrals of the motion for* $\mathcal{H}$ *are*

$$C^{(m)} = \sum_{1\leqslant i<j<k}^{m} (\lambda_i(p_j q_k - p_k q_j) + \lambda_j(p_k q_i - p_i q_k) + \lambda_k(p_i q_j - p_j q_i))^2, \tag{2.3}$$

where $m = 3, \ldots, N$*. These integrals are in involution and can be called 'universal' in the sense that they do not depend on the specific choice of the function* $\mathcal{H}$*.*

Proof. The keystone to prove this result comes from the fact that (1.1) is a one-particle symplectic realization for the Poisson coalgebra (h_6, Δ), labelled by the λ_1 parameter. Moreover, it can be easily checked that (2.2) is just the N-particle symplectic realization of (h_6, Δ) that is obtained through the N-sites generalization of the coproduct (1.5):

$$\begin{aligned} \Delta^{(N)}(X) = {} & X \otimes 1 \otimes 1 \otimes \cdots^{N-1)} \otimes 1 + 1 \otimes X \otimes 1 \otimes \cdots^{N-2)} \otimes 1 + \cdots \\ & + 1 \otimes 1 \otimes \cdots^{N-1)} \otimes 1 \otimes X. \end{aligned} \tag{2.4}$$

This means that the N-particle generators (2.2) fulfil the commutation rules (1.2) with respect to the canonical Poisson bracket $\{f, g\} = \sum_{i=1}^N \left(\frac{\partial f}{\partial q_i} \frac{\partial g}{\partial p_i} - \frac{\partial g}{\partial q_i} \frac{\partial f}{\partial p_i} \right)$.

Moreover, such a coalgebra symmetry expressed through the symplectic realization implies that each of the N-particle generators (2.2) Poisson commute with the $(N-1)$ functions $C^{(m)}$ given by the mth coproducts of the Casimir (1.4) with $m = 2, 3, \ldots, N$ (see [24, 33] for details). However, in the case of the specific symplectic realization of h_6 given by (2.2), the $C^{(2)}$ function vanishes (the two-body coproduct of the Casimir is zero [33]) and we are left with the set of $(N-2)$ integrals (2.3) that, also by construction, are functionally independent and Poisson commuting. Therefore, any function $\mathcal{H}$ (2.1) of the N-particle symplectic realization of the h_6 generators will be in involution with the set of integrals $C^{(m)}$, which completes the proof. □

This quite general result deserves the following remarks and comments.

- We can properly say that $\mathcal{H}$ is a *quasi-integrable* Hamiltonian, since for *any dimension N* and *any choice of* $\mathcal{H}$ there is only one integral left in order to get its complete integrability. Obviously, some specific choices for $\mathcal{H}$ will lead to *completely integrable* Hamiltonians for which an additional integral does exist for any dimension N. The aim of this paper is just to find solutions to this problem.

- Within the coalgebra approach it is well known that, in general, two different sets of integrals of the motion coming from 'left' and 'right' mth coproducts of the Casimir can be obtained (see [26] for details). Indeed, this is also the case for the (h_6, Δ) coalgebra, where by making use of the 'right' mth coproducts, the following alternative set of $(N-2)$ integrals in involution $C_{(m)}$ is obtained:

$$C_{(m)} = \sum_{N-m+1\leqslant i<j<k}^{N} (\lambda_i(p_jq_k - p_kq_j) + \lambda_j(p_kq_i - p_iq_k) + \lambda_k(p_iq_j - p_jq_i))^2. \qquad (2.5)$$

This means that if we label the N sites on $h_6 \otimes h_6 \otimes \cdots^{N)} \otimes h_6$ by $1\otimes 2\otimes \ldots \otimes N$, the 'left' Casimir $C^{(m)}$ is defined on the sites $1\otimes 2\otimes \ldots \otimes m$, while the 'right' one $C_{(m)}$ is defined on $(N-m+1)\otimes \ldots \otimes (N-1)\otimes N$. Moreover, it is straightforward to prove that the $(2N-4)$ functions $\{C^{(3)}, C^{(4)}, \ldots, C^{(N)} \equiv C_{(N)}, C_{(N-1)}, \ldots C_{(3)}, \mathcal{H}\}$ are functionally independent (assuming that $\mathcal{H}$ is not a function of $\mathcal{C}$ only) and the coalgebra symmetry ensures that each of the two subsets $\{C^{(3)}, \ldots, C^{(N)}, \mathcal{H}\}$ and $\{C_{(3)}, \ldots, C_{(N)}, \mathcal{H}\}$ is formed by $(N-1)$ functions in involution [24, 26].
- As a consequence, in the case that an additional integral $\mathcal{I}$ is found for a given $\mathcal{H}$, this Hamiltonian will be not only integrable, but also *superintegrable* provided that the $(N-3)$ 'right' constants $C_{(m)}$ (with $m = 3, \ldots, N-1$) commute with $\mathcal{I}$ (and this property will be ensured if $\mathcal{I}$ is a function of the N-particle symplectic realization of the h_6 coalgebra).
- The role of the λ_i parameters is essential in this approach, since they provide an N-parameter freedom for the Hamiltonian. From a coalgebraic viewpoint, these λ_i parameters can be neatly interpreted: each of them fixes the one-particle symplectic realization that we use on the ith site of the underlying h_6 symmetry lattice $h_6 \otimes h_6 \otimes \cdots^{N)} \otimes h_6$.
- We stress that the integrals (2.3) can be interpreted as sums of the squares of a linear combination (through the λ_i parameters) of 'Euclidean angular momentum' components J_{ij}. In particular, the $N(N-1)/2$ functions $J_{ij} = q_ip_j - q_jp_i$ with $i < j$ and $i, j = 1, \ldots, N$ span an $so(N)$ Lie–Poisson algebra so that $C^{(m)}$ can be read as

$$C^{(m)} = \sum_{1\leqslant i<j<k}^{m} (\lambda_i J_{jk} + \lambda_j J_{ki} + \lambda_k J_{ij})^2,$$

where $J_{ki} = -J_{ik}$. Hence, each term in $C^{(m)}$ is the square of an element of the Lie–Poisson algebra $so(3) = \{J_{ij}, J_{ik}, J_{jk}\}$ (and the same happens in $C_{(m)}$). From this perspective, the h_6-coalgebra symmetry can be interpreted as a 'generalization' of the spherical symmetry, which will be fully recovered when the Hamiltonian is defined on the $gl(2)$ Poisson subalgebra of h_6.
- Although the central generator M is also a Casimir for the two-photon coalgebra, its Nth coproduct gives rise to N trivial integrals of the motion:

$$M^{(m)} = \sum_{i=1}^{m} \lambda_i^2, \qquad m = 1, \ldots, N, \qquad (2.6)$$

that do not provide any dynamical information. In this sense, M can be considered either as a generator in its own right or as a constant that depends on both the dimension N and the chosen symplectic realizations through the λ_i parameters.

Finally, is worth mentioning that the role of the h_6 algebra in the integrability properties of certain 3D Hamiltonian systems was already pointed out from a different viewpoint in [37]. Nevertheless, the introduction of a coalgebra structure in h_6 turns out to be essential in order to fully exploit its integrability information and to generalize it to arbitrary dimensions.

2.1. Some relevant quasi-integrable Hamiltonians

Among the bunch of *N*D quasi-integrable systems that are provided by theorem 1, the following particular subclasses are physically outstanding.

2.1.1. Natural systems. The Hamiltonian

$$\mathcal{H} = \tfrac{1}{2}B_+ + \mathcal{F}(A_-, B_-), \tag{2.7}$$

where $\mathcal{F}$ is a function playing the role of a potential, gives rise to the following quasi-integrable system on the *N*D Euclidean space $\mathbb{E}^N$:

$$\mathcal{H} = \sum_{i=1}^{N} \frac{p_i^2}{2} + \mathcal{F}\left(\sum_{i=1}^{N} \lambda_i q_i, \sum_{i=1}^{N} q_i^2\right). \tag{2.8}$$

Note that central potentials (endowed with spherical symmetry) directly arise whenever $\mathcal{F}$ does not depend on A_-, since the Hamiltonian is then defined on the $gl(2)$ subalgebra. Thus in the case with generic $\mathcal{F}(A_-, B_-)$, the spherical symmetry is broken and its associated (super)integrability is, in principle, reduced to quasi-integrability. Nevertheless, the h_6-coalgebra symmetry of (2.8) reduces the *N*D integrability problem to the search for only one additional integral, whose existence and explicit form will depend on the particular choice of the function $\mathcal{F}$.

2.1.2. Electromagnetic Hamiltonians. The most general *N*D quasi-integrable Hamiltonian including linear terms in the momenta is given by

$$\mathcal{H} = \tfrac{1}{2}B_+ + K\mathcal{F}(A_-, B_-) + A_+\mathcal{G}(A_-, B_-) + \mathcal{R}(A_-, B_-), \tag{2.9}$$

where $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{R}$ are smooth functions. In terms of canonical variables, it reads as

$$\begin{aligned}\mathcal{H} = \sum_{i=1}^{N} \frac{p_i^2}{2} &+ \left(\sum_{i=1}^{N}\left(q_i p_i - \frac{\lambda_i^2}{2}\right)\right)\mathcal{F}\left(\sum_{i=1}^{N} \lambda_i q_i, \sum_{i=1}^{N} q_i^2\right) \\ &+ \left(\sum_{i=1}^{N} \lambda_i p_i\right)\mathcal{G}\left(\sum_{i=1}^{N} \lambda_i q_i, \sum_{i=1}^{N} q_i^2\right) + \mathcal{R}\left(\sum_{i=1}^{N} \lambda_i q_i, \sum_{i=1}^{N} q_i^2\right).\end{aligned} \tag{2.10}$$

In 3D, this Hamiltonian describes the motion of a particle on $\mathbb{E}^3$ under the action of a static electromagnetic field which is determined by the functions $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{R}$. Namely, if we compare (2.10) with the 3D electromagnetic Hamiltonian

$$\mathcal{H}_{em} = \tfrac{1}{2}(\vec{p} - e\vec{A})^2 + e\psi \tag{2.11}$$

where e is the electric charge, $\vec{A}$ is the vector potential and ψ is the scalar one, we get

$$A_i = -\frac{q_i}{e}\mathcal{F}(A_-, B_-) - \frac{\lambda_i}{e}\mathcal{G}(A_-, B_-), \qquad i = 1, 2, 3 \tag{2.12}$$

$$\begin{aligned}\psi = \frac{1}{e}\mathcal{R}(A_-, B_-) - \frac{1}{2e}M\mathcal{F}(A_-, B_-) - \frac{1}{2e}[B_-\mathcal{F}(A_-, B_-)^2 \\ + 2A_-\mathcal{F}(A_-, B_-)\mathcal{G}(A_-, B_-) + M\mathcal{G}(A_-, B_-)^2].\end{aligned} \tag{2.13}$$

Note the relevant role that the λ_i parameters play in the definition of the electromagnetic field. Also recall that *N*D superintegrable electromagnetic systems have been recently obtained in [36] by making use of an $sl(2, \mathbb{R})$-coalgebra symmetry. In fact, as we shall show in section 4, the latter systems are a particular subfamily of (2.10) since $sl(2, \mathbb{R})$ is a sub-coalgebra of h_6. We also recall that only low-dimensional integrable Euclidean Hamiltonians with velocity-dependent potentials have been previously studied in [38–42].

2.1.3. Geodesic flow Hamiltonians. A third family of relevant systems is that given by *N*D quasi-integrable Hamiltonians of the type

$$\mathcal{H} = \sum_{i,j=1}^{N} g^{ij}(q_1, \ldots, q_N) p_i p_j \tag{2.14}$$

that are obtained by considering

$$\begin{aligned}\mathcal{H} = {} & B_+\mathcal{F}(A_-, B_-) + A_+^2\mathcal{G}(A_-, B_-) \\ & + \left(K + \frac{M}{2}\right)^2 \mathcal{R}(A_-, B_-) + \left(K + \frac{M}{2}\right) A_+\mathcal{S}(A_-, B_-)\end{aligned} \tag{2.15}$$

since for any choice of the functions $\mathcal{F}, \mathcal{G}, \mathcal{R}$ and $\mathcal{S}$, we obtain a Hamiltonian which is a quadratic homogeneous function in the momenta. Explicitly,

$$\begin{aligned}\mathcal{H} = {} & \left(\sum_{i=1}^N p_i^2\right) \mathcal{F}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right) + \left(\sum_{i=1}^N \lambda_i p_i\right)^2 \mathcal{G}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right) \\ & + \left(\sum_{i=1}^N q_i p_i\right)^2 \mathcal{R}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right) \\ & + \left(\sum_{i=1}^N q_i p_i\right)\left(\sum_{i=1}^N \lambda_i p_i\right) \mathcal{S}\left(\sum_{i=1}^N \lambda_i q_i, \sum_{i=1}^N q_i^2\right).\end{aligned} \tag{2.16}$$

We stress that the specific form of the metric g^{ij} is determined by the $\mathcal{F}, \mathcal{G}, \mathcal{R}$ and $\mathcal{S}$ functions which, in general, give rise to an *N*D space of *nonconstant* curvature. In any case, the set of constants of motion (2.3) is universal and does not depend on the specific choice of the functions in the Hamiltonian. Moreover, additional potentials on these h_6-coalgebra spaces can be naturally considered by adding functions such as, e.g., $\mathcal{U}(A_-, B_-)$ to the free Hamiltonian (2.15). In this way, the Euclidean natural systems (2.7) can be generalized to the curved spaces defined through (2.15) without breaking the quasi-integrability of the geodesic flow Hamiltonian.

In this respect, we recall that the complete integrability of a free Hamiltonian on a curved space is a rather non-trivial property which is connected with geometric and topological features of the underlying manifold [43–46]. From the physical viewpoint, the study of integrable geodesic flows in arbitrary dimension is becoming increasingly popular in general relativity, supergravity and superstring theories, where the explicit knowledge of the Stäckel–Killing integrals of motion for the geodesic flows of *N*D curved spaces provides very useful information (see, for instance, [47–50]). Since integrable examples of such *N*D curved geodesic flows are quite scarce, the search for new completely integrable instances among the Hamiltonians (2.16) is thus meaningful.

3. Complete integrability

At this point the main problem to be faced is the characterization of those Hamiltonians $\mathcal{H}$ for which an additional integral $\mathcal{I}$ does exist *for any dimension N*, thus providing their complete *N*D integrability.

In order to ensure the existence of $\mathcal{I}$ for any dimension N, we shall assume that this additional integral is also h_6-coalgebra invariant, which means that it can be written as a function

$$\mathcal{I} = \mathcal{I}(K, B_+, B_-, A_+, A_-, M), \tag{3.1}$$

where the h_6 generators are written in their N-particle symplectic realization (2.2). In this way, if $\mathcal{I}$ is functionally independent with respect to both the h_6 Casimir (1.4) and the Hamiltonian $\mathcal{H}$, the coalgebra symmetry ensures—by construction—the involutivity of $\mathcal{I}$ with respect to the $(N-2)$ 'left' integrals $C^{(m)} (m = 3,\ldots,N)$ and its functional independence with respect to them. The very same result holds for the $(N-3)$ 'right' integrals $C_{(m)}$, where $m = 3,\ldots,N-1$ (we recall that $C_{(N)} = C^{(N)}$).

This means that if $\mathcal{I}$ does exist in form (3.1), then $\mathcal{H}$ will be not only a completely integrable system but also a *superintegrable* one, since a total number of $(N-2)+(N-3)+1 = (2N-4)$ functionally independent constants of motion for $\mathcal{H}$ has been explicitly found. Nevertheless, even in this superintegrable case, $\mathcal{H}$ is not a *maximally superintegrable* Hamiltonian, since *two* more independent constants of the motion would be needed to get the maximum possible total number of $(2N-2)$ independent integrals. Again, these two remaining integrals could exist for some very particular choices for $\mathcal{H}$, but in any case neither their existence nor their explicit form can be derived from the h_6-coalgebra symmetry.

The rest of the paper is devoted to showing how the search for the additional $\mathcal{I}$ (3.1) can be guided by taking into account the subalgebra structure of h_6. In particular, we shall consider two different situations in which the existence of $\mathcal{I}$ is guaranteed by construction.

(A) If the Hamiltonian $\mathcal{H}$ is defined within a subalgebra of h_6 that has a nonlinear Casimir invariant, the N-particle realization of the Casimir of the subalgebra provides the integral $\mathcal{I}$. This *subalgebra integrability* approach will be analysed in the following section, in which the subalgebra structure for h_6 will be fully described.

(B) Let X be a fixed generator of h_6. The N-particle symplectic realization of X will Poisson commute with any N-particle Hamiltonian $\mathcal{H}_X$ defined as a function of all the remaining h_6 generators commuting with X and of the Casimirs of all the subalgebras containing the given generator X. Under such hypotheses, $\mathcal{H}_X$ is completely integrable since the generator X is just the additional constant of motion $\mathcal{I}$. We have five relevant generators $\{K, B_+, B_-, A_+, A_-\}$ (the central generator M would give no dynamical information), so this *generator integrability* procedure will give rise to five families of completely integrable systems that will be studied in detail in section 5.

Finally, we stress that if a given Hamiltonian does not fit within the two previous approaches, the search for the remaining integral $\mathcal{I}$—in case it does exist—has to be performed by using direct methods. Indeed, some particular solutions can be found, and a particular example will be given in the concluding section.

4. Subalgebra integrability

The subalgebras of h_6 with a non-trivial (i.e. linear) Casimir function are summarized in table 1 together with their 1D symplectic realization. They are as follows.

(i) Two 'book' algebras $\mathcal{D}_+$ and $\mathcal{D}_-$ generated by a dilation plus two translations.

(ii) The harmonic oscillator algebra h_4.

(iii) Two centrally extended (1 + 1)D Galilean algebras $\overline{\mathcal{G}}_+$ and $\overline{\mathcal{G}}_-$.

(iv) A centrally extended 2D Euclidean algebra $\overline{\mathcal{E}}$ (where μ and ν are non-zero real parameters).

(v) The $gl(2)$ algebra.

More details on these subalgebras and on their associated Lie–Poisson structures can be found in [30]. Clearly the Heisenberg–Weyl algebra $h_3 = \{A_+, A_-, M\}$ is a subalgebra of

Table 1. Relevant subalgebras of h_6.

Subalgebra	Generators	Symplectic realization	Casimir function
$\mathcal{D}_+$	K, A_+, B_+	$qp - \frac{\lambda^2}{2}, \lambda p, p^2$	A_+^2/B_+
$\mathcal{D}_-$	K, A_-, B_-	$qp - \frac{\lambda^2}{2}, \lambda q, q^2$	A_-^2/B_-
h_4	K, A_-, A_+, M	$qp - \frac{\lambda^2}{2}, \lambda q, \lambda p, \lambda^2$	$M\left(K + \frac{1}{2}M\right) - A_-A_+$
$\overline{\mathcal{G}}_+$	B_+, A_-, A_+, M	$p^2, \lambda q, \lambda p, \lambda^2$	$MB_+ - A_+^2$
$\overline{\mathcal{G}}_-$	B_-, A_-, A_+, M	$q^2, \lambda q, \lambda p, \lambda^2$	$MB_- - A_-^2$
$\overline{\mathcal{E}}$	$\mu B_+ + \nu B_-,$	$\mu p^2 + \nu q^2,$	$M(\mu B_+ + \nu B_-)$
	A_-, A_+, M	$\lambda q, \lambda p, \lambda^2$	$-\mu A_+^2 - \nu A_-^2$
$gl(2)$	K, B_-, B_+, M	$qp - \frac{\lambda^2}{2}, q^2, p^2, \lambda^2$	$B_-B_+ - \left(K + \frac{1}{2}M\right)^2$

h_4, and $gl(2)$ contains an $sl(2,\mathbb{R})$ subalgebra (by mapping $K \to K + M/2$), so we have the following subalgebra embeddings:

$$h_3 \subset h_4 \subset h_6 \qquad \overline{\mathcal{G}}_\pm \subset h_6 \qquad sl(2,\mathbb{R}) \subset gl(2) \subset h_6. \tag{4.1}$$

Also note that $\overline{\mathcal{E}}$ is a proper Euclidean subalgebra whenever μ and ν have the same sign; in contrast, $\overline{\mathcal{E}}$ is in fact a centrally extended (1 + 1)D Poincaré subalgebra. In the following, we do not distinguish the two real forms as the resulting expressions for $\overline{\mathcal{E}}$ will be globally parametrized through μ and ν.

As we have pointed out in the previous section, any Hamiltonian $\mathcal{H}_g$ defined on one of the above-mentioned subalgebras g is completely integrable by construction, since the Nth coproduct of the Casimir $\mathcal{C}_g$ provides the extra integral $\mathcal{I}$, which completes the set of $(N-2)$ left integrals $C^{(m)}$ coming from the h_6 coalgebra. Note that $\mathcal{I}$ is a function of the h_6 generators and, as a consequence, is in involution with each of the $C^{(m)}$ integrals. Therefore, we can state that the following ND Hamiltonians define completely integrable systems:

$$\begin{aligned} \mathcal{H}_{\mathcal{D}_+} &= \mathcal{H}_{\mathcal{D}_+}(K, A_+, B_+) \\ \mathcal{H}_{\mathcal{D}_-} &= \mathcal{H}_{\mathcal{D}_-}(K, A_-, B_-) \\ \mathcal{H}_{h_4} &= \mathcal{H}_{h_4}(K, A_-, A_+, M) \\ \mathcal{H}_{\overline{\mathcal{G}}_+} &= \mathcal{H}_{\overline{\mathcal{G}}_+}(B_+, A_-, A_+, M) \\ \mathcal{H}_{\overline{\mathcal{G}}_-} &= \mathcal{H}_{\overline{\mathcal{G}}_-}(B_-, A_-, A_+, M) \\ \mathcal{H}_{\overline{\mathcal{E}}} &= \mathcal{H}_{\overline{\mathcal{E}}}(\mu B_+ + \nu B_-, A_-, A_+, M) \\ \mathcal{H}_{gl(2)} &= \mathcal{H}_{gl(2)}(K, B_-, B_+, M), \end{aligned} \tag{4.2}$$

where the h_6 generators are taken through their ND symplectic realization (2.2) and the Hamiltonian functions are any smooth functions of the corresponding arguments. Moreover, all these systems are superintegrable, since the h_6 coalgebra provides $(N-3)$ additional and functionally independent integrals of the motion given by the 'right' integrals $C_{(m)}$.

The explicit form of the Casimir invariant $\mathcal{C}_g \equiv \mathcal{I}$ of each subalgebra g is given in table 2 in terms of the N-particle symplectic realization (2.2). It is interesting to remark that the two-photon Casimir (1.4) can be expressed in terms of the Casimirs of the four subalgebras $h_3, h_4, \overline{\mathcal{G}}_+, \overline{\mathcal{G}}_-$ as

$$\mathcal{C}_{h_6} = \frac{1}{\mathcal{C}_{h_3}}\left(\mathcal{C}_{\overline{\mathcal{G}}_+}\mathcal{C}_{\overline{\mathcal{G}}_-} - \mathcal{C}_{h_4}^2\right). \tag{4.3}$$

Let us finally comment that all of these subalgebras are also sub-coalgebras since the same primitive coproduct (1.5) holds for all of them as Lie–Poisson algebras. In fact, an

Table 2. *N*D symplectic realization of the Casimir of each of the sub-coalgebras of (h_6, Δ) given in table 1.

Sub-coalgebra	Integrals of motion
$(\mathcal{D}_+, \Delta)$	$C^{(N)}_{\mathcal{D}_+} = \left(\sum_{i=1}^{N} \lambda_i p_i\right)^2 \Big/ \left(\sum_{j=1}^{N} p_j^2\right)$
$(\mathcal{D}_-, \Delta)$	$C^{(N)}_{\mathcal{D}_-} = \left(\sum_{i=1}^{N} \lambda_i q_i\right)^2 \Big/ \left(\sum_{j=1}^{N} q_j^2\right)$
(h_4, Δ)	$C^{(N)}_{h_4} = \sum_{1\leqslant i<j}^{N} (\lambda_j p_i - \lambda_i p_j)(\lambda_j q_i - \lambda_i q_j)$
$(\overline{\mathcal{G}}_+, \Delta)$	$C^{(N)}_{\overline{\mathcal{G}}_+} = \sum_{1\leqslant i<j}^{N} (\lambda_j p_i - \lambda_i p_j)^2$
$(\overline{\mathcal{G}}_-, \Delta)$	$C^{(N)}_{\overline{\mathcal{G}}_-} = \sum_{1\leqslant i<j}^{N} (\lambda_j q_i - \lambda_i q_j)^2$
$(\overline{\mathcal{E}}, \Delta)$	$C^{(N)}_{\overline{\mathcal{E}}} = \sum_{1\leqslant i<j}^{N} \{\mu(\lambda_j p_i - \lambda_i p_j)^2 + \nu(\lambda_j q_i - \lambda_i q_j)^2\}$
$(gl(2), \Delta)$	$C^{(N)}_{gl(2)} = \sum_{1\leqslant i<j}^{N} (q_j p_i - q_i p_j)^2$

alternative approach to the integrability of the systems (4.2) would be to consider directly the coalgebra construction for the subalgebra in which $\mathcal{H}_g$ is defined, thus forgetting about the whole h_6 scheme. In that case, the integrals of motion would be given by the 'left' and 'right' mth coproducts of the Casimir of the subalgebra $\mathcal{C}_g$, say $C_g^{(m)}$ and $C_{g,(m)}$ $(m = 2, \dots, N)$, respectively.

In this way, by taking into account that $C_g^{(2)}$ and $C_{g,(2)}$ do not vanish in the subalgebra, we would obtain (in principle) a maximum number of $(2N - 3)$ independent constants of motion for $\mathcal{H}_g$, and each set of N functions $\left\{C_g^{(m)}, \mathcal{H}_g\right\}$ or $\{C_{g,(m)}, \mathcal{H}_g\}$ would be in involution. However, in the case of $\mathcal{D}_+$, $\mathcal{D}_-$ and the two centrally extended (1 + 1)D Galilean algebras $\overline{\mathcal{G}}_+$ and $\overline{\mathcal{G}}_-$, the right integrals $C_{g,(m)}$ turn out to be functionally dependent with respect to the left ones, and in these cases the superintegrability of the associated systems can only be derived by making use of the h_6-coalgebra construction.

Some of these subalgebras have been considered previously from the coalgebra method, so that we refer to the various papers on the subject [27, 34, 36]. Nevertheless, it is worth pointing out that in the present h_6 framework the integrals of motion coming from $gl(2)$ just provide those coming from the spherical symmetry (see table 2). In terms of the $so(N)$ generators J_{ij} (see section 2), these read as

$$C^{(m)}_{gl(2)} = \sum_{1\leqslant i<j}^{m} J_{ij}^2 \qquad C_{(m),gl(2)} = \sum_{N-m+1\leqslant i<j}^{m} J_{ij}^2.$$

Therefore when a Hamiltonian of the type $\mathcal{H}_{gl(2)}$ is considered, the spherical symmetry and its associated superintegrability are recovered as a particular case of the more general h_6-coalgebra setting.

5. Generator integrability

Now, let us choose a given generator X of h_6. If we look for all the generators X_j $(j = 1, \ldots, l)$ commuting with X and we look for all the subalgebras g_k $(k = 1, \ldots, t)$ containing X as a generator, it becomes obvious that the Hamiltonian constructed through any function of the type

$$\mathcal{H}_X = \mathcal{H}_X(\mathcal{C}_{g_1}, \ldots, \mathcal{C}_{g_t}, X, X_1, \ldots, X_l), \tag{5.1}$$

where $\mathcal{C}_{g_k}$ is the Casimir function of the subalgebra g_k, verifies that

$$\{\mathcal{H}_X, X\} = 0. \tag{5.2}$$

Moreover, the Nth particle symplectic realization of both X and $\mathcal{H}_X$ will Poisson commute with the two sets of integrals $C^{(m)}$ (2.3) and $C_{(m)}$ (2.5) respectively. Therefore, the Nth symplectic realization of $\mathcal{H}_X$ is a completely integrable ND Hamiltonian system (in fact, superintegrable with a total number of $(2N-4)$ integrals of the motion).

As we shall see in the following, by taking into account the information concerning the h_6 subalgebras that is contained in tables 1 and 2, this result provides in a straightforward way a bunch of new ND integrable systems. We stress that particular integrable systems belonging to the three classes of the generically quasi-integrable Hamiltonians (2.8), (2.10) and (2.16) can be straightforwardly identified.

5.1. Hamiltonians in involution with K

Let us start by considering $X \equiv K$. It can immediately be checked that the only generator that Poisson commutes with K is M, the central one. On the other hand, K is contained in the subalgebras $\mathcal{D}_+, \mathcal{D}_-, h_4$ and $gl(2)$. Therefore, the most general Hamiltonian with the h_6-coalgebra symmetry and in involution with K is

$$\mathcal{H}_K = \mathcal{H}_K(\mathcal{C}_{D_+}, \mathcal{C}_{D_-}, \mathcal{C}_{h_4}, \mathcal{C}_{gl(2)}, K, M), \tag{5.3}$$

where the smooth function $\mathcal{H}_K : \mathbb{R}^6 \to \mathbb{R}$. Now, if we take the ND symplectic realizations of $\mathcal{H}_K$, we obtain an ND integrable Hamiltonian with $(N-1)$ integrals of motion in involution given by (2.3) together with

$$\mathcal{I} \equiv K = \sum_{i=1}^{N} q_i p_i. \tag{5.4}$$

Next, in order to classify the type of Hamiltonian systems that can be constructed from $\mathcal{H}_K$, we have to realize that the symplectic realizations of K and $\mathcal{C}_{h_4}$ are linear in the momenta, while $\mathcal{C}_{gl(2)}$ is quadratic in p. On the other hand, $\mathcal{C}_{D_+}$ is rational in p (for arbitrary λ_i) and $\mathcal{C}_{D_-}$ is a rational function in the canonical coordinates q. With these ingredients in mind and by considering the three families of systems given in section 2.1, a family of *completely integrable geodesic flows* on ND curved spaces is obtained through a choice of the Hamiltonian $\mathcal{H}_K$ leading to a quadratic homogeneous function in the momenta. Namely, the most general possibility of this type turns out to be

$$\mathcal{H}_K = \mathcal{C}_{gl(2)}\mathcal{F}(\mathcal{C}_{D_-}) + \left(K + \frac{M}{2}\right)^2 \mathcal{G}(\mathcal{C}_{D_-}) + \mathcal{C}_{h_4}^2\mathcal{R}(\mathcal{C}_{D_-}) + \left(K + \frac{M}{2}\right)\mathcal{C}_{h_4}\mathcal{S}(\mathcal{C}_{D_-}), \tag{5.5}$$

where $\mathcal{F}, \mathcal{G}, \mathcal{R}$ and $\mathcal{S}$ are smooth functions. When this Hamiltonian is written in terms of canonical coordinates, we get

$$\begin{aligned}\mathcal{H}_K = {} & \left(\sum_{1\leqslant i<j}^{N}(q_j p_i - q_i p_j)^2\right)\mathcal{F}\left(\frac{\left(\sum_{i=1}^{N}\lambda_i q_i\right)^2}{\sum_{i=1}^{N} q_i^2}\right) + \left(\sum_{i=1}^{N} q_i p_i\right)^2 \mathcal{G}\left(\frac{\left(\sum_{i=1}^{N}\lambda_i q_i\right)^2}{\sum_{i=1}^{N} q_i^2}\right)\\ & + \left(\sum_{1\leqslant i<j}^{N}(\lambda_j p_i - \lambda_i p_j)(\lambda_j q_i - \lambda_i q_j)\right)^2 \mathcal{R}\left(\frac{\left(\sum_{i=1}^{N}\lambda_i q_i\right)^2}{\sum_{i=1}^{N} q_i^2}\right)\\ & + \left(\sum_{1\leqslant i<j}^{N}(\lambda_j p_i - \lambda_i p_j)(\lambda_j q_i - \lambda_i q_j)\right)\left(\sum_{i=1}^{N} q_i p_i\right)\mathcal{S}\left(\frac{\left(\sum_{i=1}^{N}\lambda_i q_i\right)^2}{\sum_{i=1}^{N} q_i^2}\right), \end{aligned} \tag{5.6}$$

which is an ND integrable geodesic flow that depends on four smooth functions and N free parameters λ_i.

5.2. Hamiltonians in involution with A_+

When the generator A_+ is considered, we find that both B_+ and M commute with it. On the other hand, A_+ belongs to the subalgebras $\mathcal{D}_+$, h_4, $\overline{\mathcal{G}}_+$, $\overline{\mathcal{G}}_-$ and $\overline{\mathcal{E}}$. Both facts lead to the completely integrable Hamiltonian

$$\mathcal{H}_{A_+} = \mathcal{H}_{A_+}(\mathcal{C}_{\mathcal{D}_+}, \mathcal{C}_{h_4}, \mathcal{C}_{\overline{\mathcal{G}}_+}, \mathcal{C}_{\overline{\mathcal{G}}_-}, \mathcal{C}_{\overline{\mathcal{E}}}, A_+, B_+, M) \tag{5.7}$$

whatever the function $\mathcal{H}_{A_+}$. In this case, the appearance of B_+ (the Euclidean kinetic energy term) allows for a wider set of possibilities. In particular, all the following types of integrable Hamiltonians can be considered as specific cases of $\mathcal{H}_{A_+}$ for which the remaining integral is $\mathcal{I} \equiv A_+ = \sum_{i=1}^{N}\lambda_i p_i$. This integral can be interpreted as a λ_i-generalization of the translational symmetry.

- *Natural Hamiltonians.* The only possibility is

$$\mathcal{H}_{A_+} = \frac{1}{2}B_+ + \mathcal{F}(\mathcal{C}_{\overline{\mathcal{G}}_-}) = \sum_{i=1}^{N}\frac{p_i^2}{2} + \mathcal{F}\left(\sum_{1\leqslant i<j}^{N}(\lambda_j q_i - \lambda_i q_j)^2\right). \tag{5.8}$$

  Note that this Hamiltonian is *not* defined within the $\overline{\mathcal{G}}_-$ subalgebra. In the $N = 2$ case, the Calogero–Moser systems [8, 9] arise as particular choices for $\mathcal{F}$.

- *Electromagnetic Hamiltonians.* We can add linear terms in the momenta to the previous Hamiltonian leading to

$$\begin{aligned}\mathcal{H}_{A_+} &= \frac{1}{2}B_+ + \mathcal{C}_{h_4}\mathcal{G}(\mathcal{C}_{\overline{\mathcal{G}}_-}) + A_+\mathcal{R}(\mathcal{C}_{\overline{\mathcal{G}}_-}) + \mathcal{F}(\mathcal{C}_{\overline{\mathcal{G}}_-})\\ &= \sum_{i=1}^{N}\frac{p_i^2}{2} + \left(\sum_{1\leqslant i<j}^{N}(\lambda_j p_i - \lambda_i p_j)(\lambda_j q_i - \lambda_i q_j)\right)\mathcal{G}\left(\sum_{1\leqslant i<j}^{N}(\lambda_j q_i - \lambda_i q_j)^2\right)\\ &\quad + \left(\sum_{i=1}^{N}\lambda_i p_i\right)\mathcal{R}\left(\sum_{1\leqslant i<j}^{N}(\lambda_j q_i - \lambda_i q_j)^2\right) + \mathcal{F}\left(\sum_{1\leqslant i<j}^{N}(\lambda_j q_i - \lambda_i q_j)^2\right). \end{aligned} \tag{5.9}$$

- *Geodesic flows.* The most general expression coming from $\mathcal{H}_{A_+}$ and with homogeneous quadratic dependence in the momenta is given by

$$
\begin{aligned}
\mathcal{H}_{A_+} &= \mathcal{C}_{h_4}^2 \mathcal{F}(\mathcal{C}_{\overline{\mathcal{G}}_-}) + \mathcal{C}_{\overline{\mathcal{G}}_+} \mathcal{G}(\mathcal{C}_{\overline{\mathcal{G}}_-}) + B_+ \mathcal{R}(\mathcal{C}_{\overline{\mathcal{G}}_-}) + A_+^2 \mathcal{S}(\mathcal{C}_{\overline{\mathcal{G}}_-}) + A_+ \mathcal{C}_{h_4} \mathcal{T}(\mathcal{C}_{\overline{\mathcal{G}}_-}) \\
&= \left(\sum_{1 \leqslant i<j}^{N} (\lambda_j p_i - \lambda_i p_j)(\lambda_j q_i - \lambda_i q_j) \right)^2 \mathcal{F}\left(\sum_{1 \leqslant i<j}^{N} (\lambda_j q_i - \lambda_i q_j)^2 \right) \\
&\quad + \left(\sum_{1 \leqslant i<j}^{N} (\lambda_j p_i - \lambda_i p_j)^2 \right) \mathcal{G}\left(\sum_{1 \leqslant i<j}^{N} (\lambda_j q_i - \lambda_i q_j)^2 \right) \\
&\quad + \left(\sum_{i=1}^{N} p_i^2 \right) \mathcal{R}\left(\sum_{1 \leqslant i<j}^{N} (\lambda_j q_i - \lambda_i q_j)^2 \right) + \left(\sum_{i=1}^{N} \lambda_i p_i \right)^2 \mathcal{S}\left(\sum_{1 \leqslant i<j}^{N} (\lambda_j q_i - \lambda_i q_j)^2 \right) \\
&\quad + \left(\sum_{i=1}^{N} \lambda_i p_i \right) \left(\sum_{1 \leqslant i<j}^{N} (\lambda_j p_i - \lambda_i p_j)(\lambda_j q_i - \lambda_i q_j) \right) \mathcal{T}\left(\sum_{1 \leqslant i<j}^{N} (\lambda_j q_i - \lambda_i q_j)^2 \right).
\end{aligned}
\tag{5.10}
$$

5.3. Hamiltonians in involution with either A_-, B_- or B_+

To end with we jointly present these three types of Hamiltonians, since all of them provide new examples of *N*D geodesic flows.

- *A_--Hamiltonians.* If we consider that $\mathcal{I} \equiv A_- = \sum_{i=1}^{N} \lambda_i q_i$, it is straightforward to prove that the most general integrable A_--Hamiltonian reads as

$$\mathcal{H}_{A_-} = \mathcal{H}_{A_-}(\mathcal{C}_{\mathcal{D}_-}, \mathcal{C}_{h_4}, \mathcal{C}_{\overline{\mathcal{G}}_+}, \mathcal{C}_{\overline{\mathcal{G}}_-}, \mathcal{C}_{\overline{\mathcal{E}}}, A_-, B_-, M). \tag{5.11}$$

  In this case, geodesic flow Hamiltonians are available through the particular choice

$$\mathcal{H}_{A_-} = \mathcal{C}_{h_4}^2 \mathcal{F}(\mathcal{C}_{\mathcal{D}_-}, \mathcal{C}_{\overline{\mathcal{G}}_-}, A_-, B_-) + \mathcal{C}_{\overline{\mathcal{G}}_+} \mathcal{G}(\mathcal{C}_{\mathcal{D}_-}, \mathcal{C}_{\overline{\mathcal{G}}_-}, A_-, B_-) \tag{5.12}$$

  and its *N*-particle symplectic realization can be immediately obtained.
- *B_--Hamiltonians.* A similar situation is encountered when $\mathcal{I} \equiv B_- = \sum_{i=1}^{N} q_i^2$ is considered. In this case, we have that

$$\mathcal{H}_{B_-} = \mathcal{H}_{B_-}(\mathcal{C}_{\mathcal{D}_-}, \mathcal{C}_{\overline{\mathcal{G}}_-}, \mathcal{C}_{gl(2)}, B_-, A_-, M) \tag{5.13}$$

  and since $\mathcal{C}_{gl(2)}$ is the only term quadratic in the momenta, we are led to the integrable geodesic flow given by

$$\mathcal{H}_{B_-} = \mathcal{C}_{gl(2)} \mathcal{F}(\mathcal{C}_{\mathcal{D}_-}, \mathcal{C}_{\overline{\mathcal{G}}_-}, B_-, A_-). \tag{5.14}$$

- *B_+-Hamiltonians.* Finally, the last possibility is given by $\mathcal{I} \equiv B_+ = \sum_{i=1}^{N} p_i^2$. Now, the most general integrable Hamiltonian is given by

$$\mathcal{H}_{B_+} = \mathcal{H}_{B_+}(\mathcal{C}_{\mathcal{D}_+}, \mathcal{C}_{\overline{\mathcal{G}}_+}, \mathcal{C}_{gl(2)}, B_+, A_+, M). \tag{5.15}$$

  All the variables for $\mathcal{H}_{B_+}$ (except *M*) depend on the momenta. Therefore, in this case the only integrable geodesic flow can be obtained through

$$\mathcal{H}_{B_+} = \alpha B_+ + \beta A_+^2 + \gamma \mathcal{C}_{gl(2)} + \delta \mathcal{C}_{\overline{\mathcal{G}}_+}, \tag{5.16}$$

  where α, β, γ and δ are constants.

6. Concluding remarks

As we mentioned in section 3, a third possibility of showing the complete integrability for a given Hamiltonian $\mathcal{H}$ (2.1) with h_6-coalgebra symmetry is the direct search for an additional integral $\mathcal{I}$, which can be assumed to be an unknown function of the h_6 generators. An example for this type of construction is given by the following geodesic flow system:

$$\mathcal{H} = B_+(\alpha_1 A_- + \alpha_2 B_- + \alpha_3), \tag{6.1}$$

where $\alpha_1, \alpha_2, \alpha_3$ are non-vanishing constants. Despite its simplicity, this quadratic h_6 Hamiltonian neither lives in any h_6 subalgebra nor can be included within the cases studied in the previous section. However, the following additional (and functionally independent) constant of motion can be found by direct computation

$$\mathcal{I} = 4\alpha_1\alpha_2 A_+ \left(K + \frac{M}{2}\right) + 4\alpha_2\alpha_3 B_+ + 4\alpha_2^2 K(K + M) - \alpha_1^2 \mathcal{C}_{\overline{\mathcal{G}}_+}. \tag{6.2}$$

This integral provides the complete integrability of the system (6.1) for any dimension. Note that in the limit $\alpha_2 \to 0$ the Hamiltonian (6.1) belongs to the subalgebra $\overline{\mathcal{G}}_+$, and in that case $\mathcal{I}$ is just the Casimir function for such a subalgebra, as it should be. We stress that this direct search for the remaining integral can be indeed very useful, since it can be quite easily computerized. In fact, the integral $\mathcal{I}$ can be searched among h_6 functions with cubic or higher dependence on the momenta (note that all the integrals that we have presented throughout the paper are, at most, quadratic in the momenta).

Moreover, nothing prevents that, although for a certain $\mathcal{H}$ defined on h_6 the additional integral does exist, such $\mathcal{I}$ cannot be written as a function (3.1) of the h_6 generators (i.e. $\mathcal{I}$ would not be coalgebra invariant). This implies that in this case all the previous methods are not applicable and the explicit form for $\mathcal{I}$ has to be found for each dimension N, which constitutes a much more cumbersome task. Nevertheless, we would like to emphasize that any h_6-coalgebra-invariant Hamiltonian $\mathcal{H}$ of form (2.1) is only one integral away from being completely integrable, and the search for the complete list of integrable choices for $\mathcal{H}$ certainly deserves further work.

On the other hand, as was pointed out from the very beginning of the coalgebra approach to integrability, the existence of the coalgebra symmetry of $\mathcal{H}$ allows for the direct construction of integrable deformations of $\mathcal{H}$ through the use of q-deformations of the underlying Poisson coalgebra [24]. In the case of h_6, quantum two-photon/Schrödinger algebras have been constructed [31, 51] and its Poisson versions could be used to provide integrable deformations [52] of some of the systems presented here. But here it is important to recall that quantum deformations do not preserve—in general—all the sub-coalgebra structures that exist in the undeformed coalgebra (see [53]). So once a given q-deformation of h_6 had been constructed, the full analysis given in this paper has to be repeated step by step in order to elucidate which integrability properties have survived under q-deformation.

Finally, we mention that some of the new families of integrable ND Hamiltonians that we have presented in this paper deserve an individual analysis of their dynamical, geometric and physical features, which we also plan to develop in the near future.

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